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### Version of attached file:

Accepted Version

### Peer-review status of attached file:

Peer-reviewed

### Citation for published item:

Broersma, Hajo and Johnson, Matthew and Paulusma, Daniel and Stewart, Iain A. (2006) 'The computational complexity of the parallel knock-out problem.', in LATIN 2006 : theoretical informatics: : 7th Latin American symposium, Valdivia, Chile, March 20-24, 2006 : proceedings. Berlin: Springer, pp. 250-261. Lecture notes in computer science. (3887).

### Further information on publisher's website:

[https://doi.org/10.1007/11682462\\_6](https://doi.org/10.1007/11682462_6)

### Publisher's copyright statement:

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# The computational complexity of the parallel knock-out problem

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**Abstract.** We consider computational complexity questions related to parallel knock-out schemes for graphs. In such schemes, in each round, each remaining vertex of a given graph eliminates exactly one of its neighbours. We show that the problem of whether, for a given graph, such a scheme can be found that eliminates every vertex is NP-complete. Moreover, we show that, for all fixed positive integers  $k \geq 2$ , the problem of whether a given graph admits a scheme in which all vertices are eliminated in at most  $k$  rounds is NP-complete. For graphs with bounded tree-width, however, both of these problems are shown to be solvable in polynomial time.

**Keywords:** parallel knock-out; graphs; computational complexity.

## 1 Introduction

In this paper, we consider *parallel knock-out schemes* for finite undirected simple graphs. These were introduced by Lampert and Slater [5]. Such a scheme proceeds in rounds: in the first round each vertex in the graph selects exactly one of its neighbours, and then all the selected vertices are eliminated simultaneously. In subsequent rounds this procedure is repeated in the subgraph induced by those vertices not yet eliminated. The scheme continues until there are no vertices left, or until an isolated vertex is obtained (since an isolated vertex will never be eliminated).

A graph is *reducible* if there exists a parallel knock-out scheme that eliminates the whole graph. The *parallel knock-out number* of a graph  $G$ , denoted by  $\text{pko}(G)$ , is the minimum number of rounds in a parallel knock-out scheme that eliminates every vertex of  $G$ . If  $G$  is not reducible, then  $\text{pko}(G) = \infty$ . Consider the following decision problem.

PARALLEL KNOCK-OUT (PKO)

*Instance:* A graph  $G$ .

*Question:* Is  $G$  reducible?

In [5], it was claimed that PKO is NP-complete even when restricted to the class of bipartite graphs. No proof was given; the reader was referred to a paper that was in preparation. Our attempts to obtain and verify this proof have been

unsuccessful. We shall obtain the result as a corollary to a stronger theorem (Theorem 1 below) by considering a related problem, which is defined for each positive integer  $k$ .

PARALLEL KNOCK-OUT ( $k$ ) (PKO( $k$ ))

*Instance:* A graph  $G$ .

*Question:* Is  $\text{pko}(G) \leq k$ ?

That there is a polynomial algorithm to decide PKO(1) follows easily from a piece of graph theory folklore (see [1] for details). Our first result classifies the complexity of PKO( $k$ ),  $k \geq 2$ .

**Theorem 1.** *For  $k \geq 2$ , PKO( $k$ ) is NP-complete even if instances are restricted to the class of bipartite graphs.*

In [1], it was shown, using a dynamic programming approach, that the parallel knock-out number for trees can be computed in polynomial time. It was asked whether this result could be extended to graphs with bounded tree-width. In our second result, we give an affirmative answer.

**Theorem 2.** *The problem PKO( $k$ ) can be solved in linear time on graphs with bounded tree-width.*

We will also show that PKO can be solved in polynomial time on graphs with bounded tree-width.

The paper is organised as follows. In the next two sections we introduce a number of definitions and simple results. In Section 4 and Section 5 are the proofs and corollaries of Theorems 1 and 2 respectively.

## 2 Preliminaries

Graphs in this paper are denoted by  $G = (V, E)$ . An edge joining vertices  $u$  and  $v$  is denoted  $uv$ . In the *null graph*,  $V = E = \emptyset$ . For graph terminology not defined below, refer to [2].

For a vertex  $u \in V$  we denote its *neighbourhood*, that is, the set of adjacent vertices, by  $N(u) = \{v \mid uv \in E\}$ . The *degree* of a vertex is the number of edges incident with it, or, equivalently, the size of its neighbourhood.

For a graph  $G$ , a *KO-selection* is a function  $f : V \rightarrow V$  with  $f(v) \in N(v)$  for all  $v \in V$ . If  $f(v) = u$ , we say that vertex  $v$  *fires at* vertex  $u$ , or that vertex  $u$  is *knocked out* by vertex  $v$ .

For a KO-selection  $f$ , we define the corresponding *KO-successor* of  $G$  as the subgraph of  $G$  that is induced by the vertices in  $V \setminus f(V)$ ; if  $H$  is the KO-successor of  $G$  we write  $G \rightsquigarrow H$ . Note that every graph without isolated vertices has at least one KO-successor. A graph  $G$  is called *KO-reducible*, if there exists a finite sequence

$$G \rightsquigarrow G_1 \rightsquigarrow G_2 \rightsquigarrow \dots \rightsquigarrow G_r,$$

where  $G_r$  is the null graph. If no such sequence exists, then  $\text{pko}(G) = \infty$ . Otherwise, the parallel knock-out number  $\text{pko}(G)$  of  $G$  is the smallest number  $r$  for which such a sequence exists. A sequence of KO-selections that transform  $G$  into the null graph is called a *KO-reduction scheme*. A single step in this sequence is called a *round* of the KO-reduction scheme. A subset of  $V$  is *knocked out* in a certain round if every vertex in the subset is knocked out in that round.

We make some simple observations that we will use later on.

**Observation 1** *Let  $G$  be a graph on at least three vertices. If  $G$  contains two vertices of degree 1 that share the same neighbour, then  $G$  is not KO-reducible.*

**Observation 2** *Let  $u_1, u_2, u_3, u_4$  be four vertices of a KO-reducible graph  $G$  such that  $N(u_2) = \{u_1, u_3\}$ ,  $N(u_3) = \{u_2, u_4\}$  and  $N(u_4) = \{u_3\}$ . If  $u_1$  is knocked out in the first round of a KO-reduction scheme, then  $u_1$  fires at  $u_2$  in the first round.*

An odd path  $u_1 u_2 \dots u_{2k+1}$  is called a *centred path* of  $G$  with *centre vertex*  $u_{k+1}$  if  $G - \{u_{k+1}\}$  contains as components the path  $u_1 u_2 \dots u_k$  and the path  $u_{k+2} u_{k+3} \dots u_{2k+1}$ .

**Observation 3** *Let  $P = u_1 u_2 \dots u_7$  be a centred path of a KO-reducible graph  $G$ . In the first round of any KO-reduction scheme  $u_1$  and  $u_2$  fire at each other,  $u_3$  fires at  $u_2$ ,  $u_6$  and  $u_7$  fire at each other,  $u_5$  fires at  $u_6$ ,  $u_4$  fires at  $u_3$  or  $u_5$ , and  $u_4$  will not be knocked out. In the second round of any KO-reduction scheme  $u_4$  and its remaining neighbour in  $P$  fire at each other.*

### 3 NP-complete problems

In this section, we consider two NP-complete problems that we will use in the proof of Theorem 1. We refer to [4] and [6] for further details.

#### DOMINATING SET (DS)

*Instance:* A graph  $G = (V, E)$  and a positive integer  $p$ .

*Question:* Does  $G$  have a *dominating set* of size at most  $p$ , that is, is there a subset  $V' \subseteq V$  such that  $|V'| \leq p$  and every vertex of  $G$  is in  $V'$  or adjacent to a vertex in  $V'$ ?

A *hypergraph*  $J = (Q, \mathcal{S})$  is a pair of sets where  $Q = \{q_1, \dots, q_m\}$  is the vertex set and  $\mathcal{S} = \{S_1, \dots, S_n\}$  is the set of *hyperedges*. Each member  $S_j$  of  $\mathcal{S}$  is a subset of  $Q$ .

#### HYPERGRAPH 2-COLOURABILITY (H2C)

*Instance:* A hypergraph  $J = (Q, \mathcal{S})$ .

*Question:* Is there a 2-colouring of  $J = (Q, \mathcal{S})$ , that is, a partition of  $Q$  into sets  $B$  and  $W$  such that, for each  $S \in \mathcal{S}$ ,  $B \cap S \neq \emptyset$  and  $W \cap S \neq \emptyset$ .

The *incidence graph*  $I$  of a hypergraph  $J = (Q, \mathcal{S})$  is a bipartite graph with vertex set  $Q \cup \mathcal{S}$  where  $(q, S)$  forms an edge if and only if  $q \in S$ .

With a hypergraph  $J = (Q, \mathcal{S})$  we can associate another hypergraph  $J' = (X, \mathcal{Z})$  called the *triple* of  $J$ ; triples of hypergraphs will play a crucial role in our NP-completeness proofs in the next section. It requires a little effort to define the vertices  $X$  and hyperedges  $\mathcal{Z}$  of the triple of  $J$ .

Recall that  $Q = \{q_1, \dots, q_m\}$  and  $\mathcal{S} = \{S_1, \dots, S_n\}$ . For  $1 \leq i \leq m$ , let  $\ell(i)$  be the number of hyperedges in  $\mathcal{S}$  that contain  $q_i$ , let  $Q_i = \{q_i^1, \dots, q_i^{\ell(i)}\}$  and let  $U_i = \{u_i^1, \dots, u_i^{\ell(i)}\}$ . The union of all such sets is the vertex set of  $J'$ , that is

$$X = \bigcup_{i=1}^m (Q_i \cup U_i).$$

Now the hyperedges:

- for  $1 \leq i \leq m$ , for  $1 \leq k \leq \ell(i)$ , let  $P_i^k = \{q_i^k, u_i^k\}$ ,
- for  $1 \leq i \leq m$ , for  $1 \leq k \leq \ell(i) - 1$ , let  $R_i^k = T_i^k = \{u_i^k, q_i^{k+1}\}$ , and
- for  $1 \leq i \leq m$ , let  $R_i^{\ell(i)} = T_i^{\ell(i)} = \{u_i^{\ell(i)}, q_i^1\}$ .

Let  $\mathcal{P}_i = \{P_i^1, \dots, P_i^{\ell(i)}\}$ ,  $\mathcal{R}_i = \{R_i^1, \dots, R_i^{\ell(i)}\}$ , and  $\mathcal{T}_i = \{T_i^1, \dots, T_i^{\ell(i)}\}$ , and let

$$\mathcal{P} = \bigcup_{i=1}^m \mathcal{P}_i, \quad \mathcal{R} = \bigcup_{i=1}^m \mathcal{R}_i, \quad \mathcal{T} = \bigcup_{i=1}^m \mathcal{T}_i.$$

For  $1 \leq j \leq n$ , there is also a hyperedge  $S'_j$ . If in  $J$ ,  $S_j$  contains  $q_i$ , then in  $J'$ ,  $S'_j$  contains a vertex of  $Q_i$ . In particular, if  $S_j$  is the  $k$ th hyperedge that contains  $q_i$  in  $J$ , then  $S'_j$  contains  $q_i^k$ . For example, if  $q_1$  is in  $S_1, S_4$  and  $S_7$  in  $J$ , then  $\ell(1) = 3$  and in  $J'$  there are vertices  $q_1^1, q_1^2, q_1^3$  with  $q_1^1 \in S'_1$ ,  $q_1^2 \in S'_4$ , and  $q_1^3 \in S'_7$ .

Let  $\mathcal{S}' = \{S'_1, \dots, S'_n\}$ . The set of hyperedges for  $J'$  is

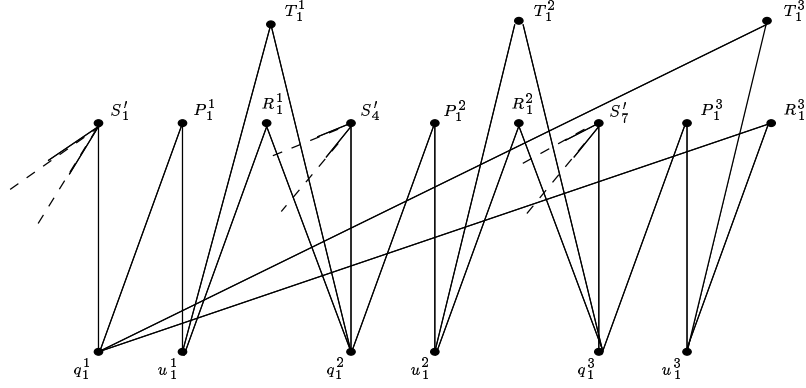
$$\mathcal{Z} = \mathcal{S}' \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{T}.$$

We denote the incidence graph of the triple  $J'$  by  $I'$ . See Figure 1 for an example that illustrates the case where  $q_1$  belongs to  $S_1, S_4$  and  $S_7$ .

**Proposition 1.**  $J = (Q, \mathcal{S})$  has a 2-colouring  $B \cup W$  if and only if  $J' = (X, \mathcal{Z})$  has a 2-colouring  $B' \cup W'$  such that for each  $1 \leq i \leq m$  either  $Q_i \subseteq B'$  and  $U_i \subseteq W'$ , or  $Q_i \subseteq W'$  and  $U_i \subseteq B'$ .

*Proof.* Suppose  $B \cup W$  is a 2-colouring of  $J$ . Define a partition  $B' \cup W'$  of  $X$  as follows. If  $q_i$  is in  $B$ , then each  $q_i^k$  is in  $B'$  and each  $u_i^k$  is in  $W'$ . If  $q_i$  is in  $W$ , then each  $q_i^k$  is in  $W'$  and each  $u_i^k$  is in  $B'$ . Obviously,  $B' \cup W'$  is a 2-colouring of  $J'$  with the desired property.

Suppose we have a 2-colouring  $B' \cup W'$  of  $J'$  such that for each  $1 \leq i \leq m$  either  $Q_i \subseteq B'$  and  $U_i \subseteq W'$ , or  $Q_i \subseteq W'$  and  $U_i \subseteq B'$ . Then let  $q_i \in B$  if and only if  $Q_i \subseteq B'$ , and let  $W = Q \setminus B$ . Clearly, if  $S_j$  contains only elements from  $B$  (respectively  $W$ ), then  $S'_j$  would contain only elements from  $B'$  (respectively  $W'$ ). Hence  $B \cup W$  is a 2-colouring of  $J$ .  $\square$



**Fig. 1.** Part of the incidence graph of the triple of a hypergraph.

## 4 Complexity classification

**Theorem 1** *For  $k \geq 2$ ,  $\text{PKO}(k)$  is NP-complete even if instances are restricted to the class of bipartite graphs.*

*Proof.* The proof is in three cases.

**Case 1.**  $k = 2$ . We use reduction from DS. Given  $G = (V, E)$  and a positive integer  $p \leq |V|$ , we shall construct a bipartite graph  $B$  such that  $\text{pko}(B) = 2$  if and only if  $G$  has a dominating set  $D$  where  $|D| \leq p$ .

Let the vertex set of  $B$  be the disjoint union of  $V = \{v_1, \dots, v_n\}$ ,  $V' = \{v'_1, \dots, v'_n\}$  and  $W = \{w_1, \dots, w_{n-p}\}$ . Let the edge set of  $B$  contain

- $v_i v'_i$ ,  $1 \leq i \leq n$ ,
- $v_i v'_j$  and  $v'_i v_j$ , for each edge  $v_i v_j \in E$ , and
- $v_i w_h$ ,  $1 \leq i \leq n$ ,  $1 \leq h \leq n - p$ .

Suppose that  $G$  has a dominating set  $D = \{v_1, \dots, v_d\}$  where  $d \leq p$ . Note that every vertex in  $V'$  is adjacent to a vertex of  $D$  in  $B$ . We shall describe a 2-round KO-reduction scheme for  $B$ . In round 1

- for  $1 \leq i \leq n$ ,  $v_i$  fires at  $v'_i$ ,
- for  $1 \leq j \leq p$ ,  $v'_j$  fires at  $v_j$ ,
- for  $p + 1 \leq j \leq n$ ,  $v'_j$  fires at a vertex in  $D$ , and
- for  $1 \leq h \leq n - p$ ,  $w_h$  fires at a vertex in  $D$ .

Thus each vertex in  $\{v_1, \dots, v_p\}$  and  $V'$  is eliminated, and each vertex in  $V \setminus \{v_1, \dots, v_p\}$  and  $W$  survives to round 2. As the surviving vertices induce the balanced complete bipartite graph  $K_{n-p, n-p}$  in  $B$ , it is clear that every surviving vertex can be eliminated in one further round.

Now suppose that  $B$  has a 2-round KO-reduction scheme. Let  $D$  be the subset of  $V$  containing vertices that are fired at in round 1. As every vertex in  $V'$  fires

at — and so is adjacent to — a vertex in  $D$ ,  $D$  is a dominating set in  $G$  (since each vertex in  $V'$  is joined only to copies of itself and its neighbours). We must show that  $|D| \leq p$ . Let  $V_S = V \setminus D$  and  $V'_S \subset V' \cup W$  be the sets of vertices that survive round 1. As round 2 is the final round,

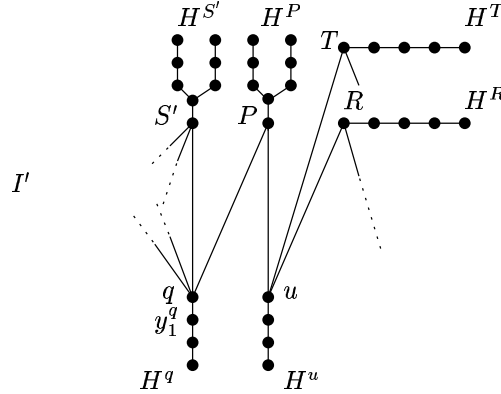
$$|V_S| = |V'_S|. \quad (1)$$

As  $|V' \cup W| = 2n - p$  and at most  $n$  vertices in  $V' \cup W$  are fired at in round 1,  $|V'_S| \geq n - p$ . Thus, by (1),  $|V_S| \geq n - p$ . Therefore

$$\begin{aligned} |D| &= |V| - |V_S| \\ &\leq n - (n - p) \\ &= p. \end{aligned}$$

**Case 2.**  $k = 3$ . Let  $J = (Q, \mathcal{S})$  be an instance of H2C. Let  $I'$  be the incidence graph of its triple  $J' = (X, \mathcal{Z})$ . Recall that  $\mathcal{Z} = \mathcal{S}' \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{T}$ . From  $I'$ , we obtain a further bipartite graph  $G$  by connecting each vertex with a path as follows:

- For each vertex  $x$  in  $X$ , w add a path  $H^x = y_1^x y_2^x y_3^x$  and join  $x$  to  $y_1^x$ .
- For each vertex  $R$  in  $\mathcal{R}$ , add a path  $H^R = y_1^R \dots y_4^R$  and join  $R$  to  $y_1^R$ .
- For each vertex  $T$  in  $\mathcal{T}$ , add a path  $H^T = y_1^T \dots y_4^T$  and join  $T$  to  $y_1^T$ .
- For each vertex  $P$  in  $\mathcal{P}$ , add a path  $H^P = y_1^P \dots y_7^P$  and join  $P$  to the centrevertex  $y_4^P$ .
- For each vertex  $S'$  in  $\mathcal{S}'$ , add a path  $H^{S'} = y_1^{S'} \dots y_7^{S'}$  and join  $S'$  to the centrevertex  $y_4^{S'}$ .



**Fig. 2.** The graph  $G$  in Case 2.

Figure 2 illustrates  $G$ . We shall prove that  $J$  is 2-colourable if and only if  $\text{pko}(G) \leq 3$ . Throughout the proof,  $G_1$  and  $G_2$  denote the graphs induced by the surviving vertices after, respectively, 1 and 2 rounds of a KO-reduction scheme.

Suppose  $B \cup W$  is a 2-colouring of  $J$ . By Proposition 1,  $J'$  has a 2-colouring  $B' \cup W'$ . We define a three-round KO-reduction scheme for  $G$ .

*Round 1.* Vertices of degree 1 and their neighbours fire at each other. Each  $H^P$  with  $P \in \mathcal{P}$  and each  $H^{S'}$  with  $S' \in \mathcal{S}'$  is a centred path of  $G$ , and the vertices fire as in Observation 3. For each  $z \in \mathcal{R} \cup \mathcal{T}$ , vertex  $y_1^z$  fires at  $y_2^z$  and  $y_2^z$  fires at  $y_3^z$ . Each vertex in  $\mathcal{Z}$  fires at one of its neighbours in  $B'$ . Each vertex  $x$  in  $X$  fires at its neighbour  $y_1^x$  in  $H^x$ . Each  $y_1^x$  with  $x \in B'$  fires at  $x$ . Each  $y_1^x$  with  $x \in W'$  fires at  $y_2^x$ .

Thus every vertex in  $W'$  and no vertex in  $B'$  survives. Also every vertex in  $\mathcal{Z}$  survives. Each vertex  $z \in \mathcal{R} \cup \mathcal{T}$  is adjacent to a vertex  $y_1^z$  of degree 1, and each vertex  $z \in \mathcal{S}' \cup \mathcal{P}$  is adjacent to a vertex  $y_4^z$  whose only other neighbour is a vertex  $y_3^z$  of degree 1.

*Round 2.* Because  $B' \cup W'$  is a 2-colouring of  $J = (X, \mathcal{Z})$ , every vertex in  $\mathcal{Z}$  has a neighbour in  $W'$  in  $G_1$ . For each  $S'_j \in \mathcal{S}'$  we choose one neighbour in  $W'$  and let  $W''$  be the set of selected vertices. Since no two vertices in  $\mathcal{S}'$  have a common neighbour in  $X$ ,  $|W''| = n$ . The vertices in  $G_1$  fire as follows. Vertices of degree 1 and their neighbours fire at each other. Each vertex  $P \in \mathcal{P}$  with a neighbour in  $W' \setminus W''$  fires at this neighbour. Otherwise  $P$  fires at  $y_4^P$ . Each  $x \in X$  fires at its neighbour in  $\mathcal{P}$ . Each  $S' \in \mathcal{S}'$  fires at  $y_4^{S'}$ .

Thus the vertex set of  $G_2$  is  $W'' \cup \mathcal{S}'$ .

*Round 3.* Each  $S' \in \mathcal{S}'$  and its unique neighbour in  $W''$  fire at each other, which leaves us with the null graph.

Now we suppose that  $\text{pko}(G) \leq 3$ . We assume that a particular KO-reduction scheme for  $G$  is given and prove that  $J$  has a 2-colouring.

*Claim 1.* If a vertex in a set  $Q_i$  is knocked out in the first round, then all vertices in  $Q_i$  are knocked out in the first round.

Suppose that vertex  $q_i^k \in Q_i$  is knocked out in the first round. We show that  $q_i^{k+1}$  (with  $q_i^{\ell(i)+1} = q_i^1$ ) is also knocked out in the first round.

If  $q_i^k \in Q_i$  is knocked out in the first round, then, by Observation 2,  $q_i^k$  fires at  $y_1^{q_i^k}$ . Suppose  $q_i^{k+1}$  is *not* knocked out in the first round. Observation 3 implies that  $P_i^{k+1}$  must fire at  $u_i^{k+1}$  and  $P_i^k$  must fire at either  $q_i^k$  or  $u_i^k$ . If  $P_i^k$  fires at  $u_i^k$ , then by Observation 2  $u_i^k$  fires at  $y_1^{q_i^k}$ . Since vertices in  $H^{P_i^k}$  must fire as in Observation 3, this means that  $G_1$  contains a component isomorphic to a path on three vertices. By Observation 1  $G_1$  is not KO-reducible. Hence,  $P_i^k$  fires at  $q_i^k$ .

For the same reason  $R_i^{k+1}$  or  $T_i^{k+1}$  cannot fire at  $u_i^k$ , and consequently, fire at  $y_1^{R_i^{k+1}}$  and  $y_1^{T_i^{k+1}}$  respectively. Due to Observation 2 this implies that  $y_1^{R_i^{k+1}}$  fires at  $y_2^{R_i^{k+1}}$ , and  $y_1^{T_i^{k+1}}$  fires at  $y_2^{T_i^{k+1}}$ .

In  $G_1$  both  $T_i^k$  and  $R_i^k$  have exactly the same neighbours, namely  $u_i^k$  and  $q_i^{k+1}$ . If  $T_i^k$  and  $R_i^k$  fire at a different neighbour in the second round, then due to Observation 2 both will be isolated vertices in  $G_2$ . Suppose  $T_i^k$  and  $R_i^k$  fire at



the same neighbour. Then in all possible schemes  $G_2$  will contain two vertices of degree 1 having the same neighbour. Observation 1 implies that  $G_2$  is not KO-reducible. We conclude that  $q_i^{k+1}$  must be knocked out in the first round as well, and this proves the claim.

*Claim 2.* If a vertex in a set  $U_i$  is knocked out in the first round, then all vertices in  $U_i$  are knocked out in the first round.

This claim is proven by using the same arguments as in Claim 1.

By Claim 1 and Claim 2 we may define a set  $B' \subseteq X$  as follows. All vertices of a set  $Q_i$  or  $U_i$  are in  $B'$  if and only if the set is knocked out in the first round. Let  $W' = X \setminus B'$ .

*Claim 3.* For all  $1 \leq i \leq m$ , either  $Q_i \subseteq B'$  and  $U_i \subseteq W'$ , or  $Q_i \subseteq W'$  and  $U_i \subseteq B'$ .

Let  $1 \leq i \leq m$ . By Observation 3, each vertex  $P_i^k \in \mathcal{P}_i$  must fire at either  $q_i^k$  or  $u_i^k$  in the first round. The previous two claims imply that  $Q_i$  or  $U_i$  is knocked out in the first round. Suppose both sets are knocked out in the first round. Then, by Observation 2,  $u_i^1$  fires at  $y_1^{u_i^1}$  and  $q_i^1$  fires at  $y_1^{q_i^1}$ . Then, by Observation 3,  $P_i^1$  will not be knocked out in any round. The claim is proved.

By Claim 3, all vertices in  $\mathcal{Z} \setminus \mathcal{S}'$  have one neighbour in  $B'$  and one neighbour in  $W'$ . Let  $S'_j$  be a vertex in  $\mathcal{S}$ . By Observation 3,  $S'_j$  fires at a neighbour in  $\bigcup_{i=1}^m Q_i$ . By definition, this neighbour is in  $B'$ . By both Observation 2 and Observation 3,  $S'_j$  is knocked out by a neighbour in  $\bigcup_{i=1}^m Q_i$  that is not knocked out in the first round. By definition, this neighbour is in  $W'$ . It is now clear that  $B' \cup W'$  is a 2-colouring of  $J'$  such that for each  $1 \leq i \leq m$  either  $Q_i \subseteq B'$  and  $U_i \subseteq W'$ , or  $Q_i \subseteq W'$  and  $U_i \subseteq B'$ . Hence, by Proposition 1,  $J$  also has a 2-colouring.

**Case 3.**  $k \geq 4$ . We use reduction from H2C. From an instance  $J = (Q, \mathcal{S})$  we construct the graph  $G$  as in the previous case. We claim that  $J$  is 2-colourable if and only if  $\text{pko}(G) \leq k$ .

Suppose that  $J$  is 2-colourable. As we have seen in the previous case this implies that  $\text{pko}(G) \leq 3 \leq k$ .

Suppose that  $\text{pko}(G) \leq k$ . Then  $G$  is KO-reducible. Note that in the proof of the previous case we only assume that  $G$  is KO-reducible. Hence we can copy the proof of the previous case. This completes the proof of Theorem 1.  $\square$

**Corollary 1.** *The PKO problem is NP-complete, even if instances are restricted to the class of bipartite graphs.*

*Proof.* We use reduction from H2C. From an instance  $J = (Q, \mathcal{S})$  we construct the graph  $G$  as in the proof of Theorem 1. We claim that  $J$  is 2-colourable if and only if  $G$  is KO-reducible.

Suppose that  $J$  is 2-colourable. As we have seen in the proof of Theorem 1 this implies that  $\text{pko}(G) \leq 3$ . Hence  $G$  is KO-reducible.

Suppose that  $G$  is KO-reducible. We copy the proof of Case 2 of Theorem 1.  $\square$

EXACT PARALLEL KNOCK-OUT ( $k$ ) (EPKO( $k$ ))

*Instance:* A graph  $G$ .

*Question:* Is  $\text{pko}(G) = k$ ?

**Corollary 2.** *The EPKO( $k$ ) problem is polynomially solvable for  $k = 1$  and is NP-complete for  $k \geq 2$ , even if instances are restricted to the class of bipartite graphs.*

*Proof.* For the case  $k = 1$  we only have to exclude the null graph. Let  $k \geq 2$ . In [1] a family of trees  $Y_\ell$  is constructed with  $\text{pko}(Y_\ell) = \ell$  for  $\ell \geq 1$ . For the case  $k = 2$  we only have to add a disjoint copy of the tree  $Y_2$  (a path on 7 vertices) to the graph  $B$  in the proof of Case 1 in Theorem 1. For  $k \geq 3$  it suffices to add a disjoint copy of the tree  $Y_k$  to the graph  $G$  constructed in the proof of Case 2 in Theorem 1. Note that the size of a tree  $Y_k$  only depends on  $k$  and not on the size of our input graph  $G$  (so we do not need the exact description of this family).  $\square$

## 5 Bounded tree-width

In this section we use *monadic second-order logic*; that is, that fragment of second-order logic where quantified relation symbols must have arity 1. For example, the following sentence, which expresses that a graph (whose edges are given by the binary relation  $E$ ) can be 3-coloured, is a sentence of monadic second-order logic:

$$\begin{aligned} \exists R \exists W \exists B \Big\{ & \forall x \left( (R(x) \vee W(x) \vee B(x)) \wedge \neg(R(x) \wedge W(x)) \right. \\ & \left. \wedge \neg(R(x) \wedge B(x)) \wedge \neg(W(x) \wedge B(x)) \right) \wedge \forall x \forall y \left( E(x, y) \Rightarrow \right. \\ & \left. \left. (\neg(R(x) \wedge R(y)) \wedge \neg(W(x) \wedge W(y)) \wedge \neg(B(x) \wedge B(y))) \right) \right\} \end{aligned}$$

(the quantified unary relation symbols are  $R$ ,  $W$  and  $B$ , and should be read as sets of ‘red’, ‘white’ and ‘blue’ vertices, respectively). Thus, in particular, there exist **NP**-complete problems that can be defined in monadic second-order logic.

A seminal result of Courcelle [3] is that on any class of graphs of bounded tree-width, every problem definable in monadic second-order logic can be solved in time linear in the number of vertices of the graph. Moreover, Courcelle’s result holds not just when graphs are given in terms of their edge relation, as in the example above, but also when the domain of a structure encoding a graph  $G$  consists of the disjoint union of the set of vertices and the set of edges, as well as unary relations  $V$  and  $E$  to distinguish the vertices and the edges, respectively, and also a binary incidence relation  $I$  which denotes when a particular vertex is incident with a particular edge (thus,  $I \subseteq V \times E$ ). The reader is referred to [3] for more details and also for the definition of tree-width which is not required here. To prove Theorem 2, we need only prove the following proposition.

**Proposition 2.** For  $k \geq 1$ ,  $\text{PKO}(k)$  can be defined in monadic second order logic.

*Proof.* Recall that a parallel knock-out scheme for a graph  $G = (V, E)$  is a sequence of graphs

$$G \rightsquigarrow G_1 \rightsquigarrow G_2 \rightsquigarrow \cdots \rightsquigarrow G_r,$$

where  $G_r$  is the null graph. Let  $W_0 = V$  and, for  $1 \leq i \leq r$ , let  $W_i$  be the vertex set of  $G_i$ . If we can write a formula  $\Phi(W_i, W_{i+1})$  of monadic second-order logic that says

*there exists a KO-selection  $f_i$  on  $W_i$  such that the vertex set of the KO-successor is  $W_{i+1}$ ,*

then we could prove the proposition with the following sentence  $\Omega_k$  which is satisfied if and only if  $G$  is in  $\text{PKO}(k)$ :

$$\begin{aligned} & \exists W_0 \exists W_1 \cdots \exists W_k (\forall v (W_0(v) \Leftrightarrow V(v)) \\ & \quad \wedge \Phi(W_0, W_1) \wedge \Phi(W_1, W_2) \wedge \cdots \wedge \Phi(W_{k-1}, W_k) \\ & \quad \wedge (\forall v (\neg W_k(v) \Leftrightarrow V(v))). \end{aligned}$$

(Here and elsewhere we have presupposed that each  $W_i$  is a set of vertices; we could easily include additional clauses to check this explicitly.)

The following claim will help us write  $\Phi(W_i, W_{i+1})$ .

*Claim 4.* There is a KO-selection  $f_i$  on  $W_i$  such that  $W_{i+1}$  is the vertex set of the KO-successor if and only if there is a partition  $V_1, V_2, V_3$  of  $W_i$  and subsets  $E_1, E_2, E_3$  of  $E$  such that

- (a) for  $j = 1, 2, 3$ , each vertex in  $V_j$  is incident with exactly one edge of  $E_j$ , this edge joins it to a vertex in  $W_i \setminus V_j$ , and this accounts for every edge in  $E_j$  (so  $|V_j| = |E_j|$ ).
- (b)  $W_{i+1} \subset W_i$  and, for  $j = 1, 2, 3$ ,  $W_{i+1} \cap V_j$  is the set of vertices in  $V_j$  not incident with edges in  $E_{j'}$  for any  $j' \neq j$ .

We will prove the claim later. First we use it to write  $\Phi(W_i, W_{i+1})$ .

The following formula  $\psi(V_1, E_1, V_2, E_2, V_3, E_3, W_i)$  checks that the sets  $V_1, V_2$  and  $V_3$  partition  $W_i$ , that the sets  $E_1, E_2, E_3$  are edges in the graph, and that (a) is satisfied.

$$\begin{aligned} & \forall v ((V_1(v) \vee V_2(v) \vee V_3(v)) \Leftrightarrow W_i(v)) \wedge \forall v (\neg(V_1(v) \wedge V_2(v)) \\ & \quad \wedge \neg(V_1(v) \wedge V_3(v)) \wedge \neg(V_2(v) \wedge V_3(v))) \\ & \quad \wedge \forall x ((E_1(x) \vee E_2(x) \vee E_3(x)) \Rightarrow E(x)) \\ & \quad \wedge \forall x (E_1(x) \Rightarrow \exists u \exists v (V_1(u) \wedge (V_2(v) \vee V_3(v)) \wedge I(u, x) \wedge I(v, x))) \\ & \quad \wedge \forall x (E_2(x) \Rightarrow \exists u \exists v (V_2(u) \wedge (V_1(v) \vee V_3(v)) \wedge I(u, x) \wedge I(v, x))) \\ & \quad \wedge \forall x (E_3(x) \Rightarrow \exists u \exists v (V_3(u) \wedge (V_1(v) \vee V_2(v)) \wedge I(u, x) \wedge I(v, x))) \\ & \quad \wedge \forall v (V_1(v) \Rightarrow \exists! x (I(v, x) \wedge E_1(x))) \\ & \quad \wedge \forall v (V_2(v) \Rightarrow \exists! x (I(v, x) \wedge E_2(x))) \\ & \quad \wedge \forall v (V_3(v) \Rightarrow \exists! x (I(v, x) \wedge E_3(x))) \end{aligned}$$

(The semantics of  $\exists!$  is ‘there exists exactly one’; clearly, this abbreviates a more complex though routine first-order formula.) The following formula checks that (b) is satisfied and is denoted  $\chi(V_1, E_1, V_2, E_2, V_3, E_3, W_i, W_{i+1})$ .

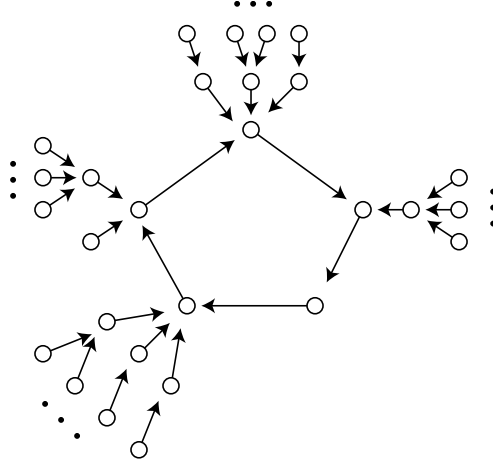
$$\begin{aligned} \forall v(W_{i+1}(v) \Leftrightarrow (W_i(v) \wedge ((V_1(v) \wedge \neg \exists x((E_2(x) \vee E_3(x)) \wedge I(v, x))) \\ \vee (V_2(v) \wedge \neg \exists x((E_1(x) \vee E_3(x)) \wedge I(v, x))) \\ \vee (V_3(v) \wedge \neg \exists x((E_1(x) \vee E_2(x)) \wedge I(v, x)))). \end{aligned}$$

And now we can write  $\Phi(W_i, W_{i+1})$ :

$$\begin{aligned} \exists V_1 \exists E_1 \exists V_2 \exists E_2 \exists V_3 \exists E_3 (\psi(V_1, E_1, V_2, E_2, V_3, E_3, W_i) \\ \wedge \chi(V_1, E_1, V_2, E_2, V_3, E_3, W_i, W_{i+1})). \end{aligned}$$

It only remains to prove Claim 4. Suppose that we have sets  $V_1, V_2, V_3, E_1, E_2$  and  $E_3$  that satisfy the conditions of the claim. Then to define the KO-selection  $f_i$ , for  $j = 1, 2, 3$ , for each vertex  $v \in V_j$ , let  $v$  fire at the unique neighbour joined to  $v$  by an edge in  $E_j$ . It is easy to check that  $W_{i+1}$  is the vertex set of the KO-successor.

Now suppose that we have a KO-selection  $f_i$ . Let  $H_i$  be the spanning subgraph of  $G_i$  with edge set  $\{vf_i(v) \mid v \in W_i\}$ . The firing can be represented as an orientation of  $H$ : orient each edge from  $v$  to  $f_i(v)$  (some edges may be oriented in both directions). As each vertex has exactly one edge oriented away from it, each component of the oriented graph contains one directed cycle, of length at least 2, with a pendant in-tree attached to each vertex of the cycle; see Figure 3.



**Fig. 3.** A representation of vertices firing

We find the sets  $V_1, V_2, V_3, E_1, E_2, E_3$ ; the edge sets contain only edges of  $H_i$ . We may assume that  $H_i$  is connected (else we can find the sets componentwise).

Let the vertices of the unique cycle in the orientation be  $v_1, \dots, v_c$  where the edges are  $v_l v_{l+1}$ ,  $1 \leq l \leq c-1$ , and  $v_c v_1$ . So  $H_i$  contains vertices  $v_1, \dots, v_c$  with a pendant tree (possibly trivial) attached to each.

For  $1 \leq l \leq c$ , let  $U_e^l$  be the set of vertices in the pendant tree attached to  $v_l$  whose distance from  $v_l$  is even (but not zero), and let  $U_o^l$  be the vertices in the tree at odd distance from  $v_l$ . Let

$$\begin{aligned} V_1 &= \bigcup_{l \text{ odd}} U_o^l \cup \bigcup_{l \text{ even}} U_e^l \cup \{v_l : l \text{ is even}, l \neq c\}, \\ V_2 &= \bigcup_{l \text{ odd}} U_e^l \cup \bigcup_{l \text{ even}} U_o^l \cup \{v_l : l \text{ is odd}, l \neq c\}, \text{ and} \\ V_3 &= \{v_c\}, \end{aligned}$$

and, for  $i = 1, 2, 3$ , let  $E_i$  contain  $vf_i(v)$  for each  $v \in V_i$ . It is clear that the sets we have chosen satisfy the conditions of the claim.

This completes the proof of the claim and of the proposition.  $\square$

Theorem 2 follows from the proposition. And, noting that  $\text{EPKO}(k)$  is defined by the monadic second-order sentence  $\Omega_k \wedge \neg \Omega_{k-1}$ , we have the following result.

**Corollary 3.** *For  $k \geq 1$ ,  $\text{EPKO}(k)$  is solvable in linear time on any class of graphs with bounded tree-width.*

Finally, we note that to check whether a graph  $G$  is reducible it is sufficient to check whether  $\text{pko}(G) = k$ , for  $1 \leq k \leq \Delta$ , where  $\Delta$  is the maximum degree of  $G$ . Thus  $G$  is reducible if and only if the sentence  $\Omega_\Delta \vee \Omega_{\Delta-1} \vee \dots \vee \Omega_1$  is satisfied. This gives us our last result.

**Corollary 4.** *On any class of graphs with bounded tree-width, PKO can be solved in polynomial time.*

## References

1. H. BROERSMA, F.V. FOMIN, R.KRÁLOVIČ, AND G.J. WOEGINGER. Eliminating graphs by means of parallel knock-out schemes, to appear in *Discrete Mathematics*.
2. J.A. BONDY, AND U.S.R. MURTY (1976). *Graph Theory with Applications*. Macmillan, London and Elsevier, New York.
3. B. COURCELLE (1990). The monadic second-order logic of graphs. I. Recognizable sets of finite graphs, *Information and Computation* 85, 12–75.
4. M.R. GAREY AND D.S. JOHNSON (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, San Francisco.
5. D.E. LAMPERT AND P.J. SLATER (1998). Parallel knockouts in the complete graph. *American Mathematical Monthly* 105, 556–558.
6. L. LOVÁSZ (1973). Covering and coloring of hypergraphs, *Proceedings of the 4th Southeastern Conference on Combinatorics, Graph Theory, and Computing*, Utilitas Mathematica, 3–12.